New Lax pair for restricted multiple three wave interaction system, quasiperiodic solutions and bi-hamiltonian structure

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(Dated: February 5, 2008)

We study restricted multiple three wave interaction system by the inverse scattering method. We develop the algebraic approach in terms of classical r-matrix and give an interpretation of the Poisson brackets as linear r-matrix algebra. The solutions are expressed in terms of polynomials of theta functions. In particular case for $n = 1$ in terms of Weierstrass functions.

I. RESTRICTED MULTIPLE THREE WAVE INTERACTION SYSTEM

Several studies have appeared recently on coupled quadratic nonlinear oscillators [1],[2],[3]

$$i\frac{db_j}{d\xi} + uc_j - \frac{1}{2}\epsilon_j b_j = 0, \tag{1}$$

$$i\frac{dc_j}{d\xi} + u^*b_j + \frac{1}{2}\epsilon_j c_j = 0, \tag{2}$$

$$i\frac{du}{d\xi} + \sum_{j=1}^{n} b_j c_j^* = 0, \tag{3}$$

where ξ is the evolution coordinate and ϵ_j corresponds to the normalized wave number mismatchs. The system (1-3) is introduced in [4] to model the growth of a low frequency internal ocean wave by interaction with higher frequency surface waves and is used in [5] as a model of plasma turbulence. This system describe triads of waves (a_j, b_j, u), $j = 1, \ldots n$ evolving in ξ and interacting with each other trough multiple three wave interaction with possible applications in optics.

II. LAX REPRESENTATION

Let us consider coupled quadratic nonlinear oscillators

$$i\frac{db_j}{d\xi} + uc_j - \frac{1}{2}\epsilon_j b_j = 0, \tag{4}$$

$$i\frac{dc_j}{d\xi} + u^*b_j + \frac{1}{2}\epsilon_j c_j = 0, \tag{5}$$

$$i\frac{du}{d\xi} + \sum_{j=1}^{n} b_j c_j^* = 0, \tag{6}$$

where ξ is the evolution coordinate and ϵ_j are constants. The equations (4-6) can be written as Lax representation

$$\frac{dL}{d\xi} = [M, L], \tag{7}$$

of the following linear system:

$$\frac{d\psi}{d\xi} = M(\xi, \lambda)\psi(\xi, \lambda) \quad L(\xi, \lambda)\psi(\xi, \lambda) = 0, \tag{8}$$

where L, M are 2×2 matrices and have the form

$$L(\xi, \lambda) = \begin{pmatrix} A(\xi, \lambda) & B(\xi, \lambda) \\ C(\xi, \lambda) & D(\xi, \lambda) \end{pmatrix}, \tag{9}$$

$$M(\xi, \lambda) = \begin{pmatrix} -\imath\lambda/2 & iu \\ u^* & \imath\lambda/2 \end{pmatrix}. \tag{10}$$

where

$$A(\xi, \lambda) = a(\lambda) \left(-\imath\frac{\lambda}{2} + \frac{\imath}{2} \sum_{j=1}^{n} \frac{(c_j c_j^* - b_j b_j^*)}{\lambda - \epsilon_j} \right), \tag{11}$$

$$B(\xi, \lambda) = a(\lambda) \left(\imath u - \imath \sum_{j=1}^{n} \frac{b_j c_j^*}{\lambda - \epsilon_j} \right), \tag{12}$$

$$C(\xi, \lambda) = a(\lambda) \left(\imath u^* - \imath \sum_{j=1}^{n} \frac{c_j b_j^*}{\lambda - \epsilon_j} \right), \tag{13}$$

where $D(\xi, \lambda) = -A(\xi, \lambda)$ and $a(\lambda) = \prod_{i=1}^{n}(\lambda - \epsilon_i)$. The Lax representation yields the hyperelliptic curve $K = (\nu, \lambda)$

$$\det(L(\lambda) - \frac{1}{2}\nu I) = 0, \tag{14}$$

where I is the 2×2 unit matrix. The moduli of the curve (14) generate the integrals of motion $J_0, J_j, K_j, j = 1, \ldots, n$,

$$\nu^2 = A^2(\xi, \lambda) + B(\xi, \lambda)C(\xi, \lambda). \tag{15}$$

The curve (15) can be written in canonical form as

$$\nu^2 = 4 \prod_{j=1}^{2n+2} (\lambda - \lambda_j) = R(\lambda), \tag{16}$$

where $\lambda_j \neq \lambda_k$ are branching points. From (15) and explicit expressions for $A(\xi, \lambda), B(\xi, \lambda), C(\xi, \lambda)$ we obtain

$$\nu^2 = a(\lambda)^2 \left(\lambda^2 - 4iJ_0 + 4i \sum_{j=1}^{n} \frac{K_j}{\lambda - \epsilon_j} - i \sum_{j=1}^{n} \frac{J_j^2}{(\lambda - \epsilon_j)^2} \right), \tag{17}$$

where

$$K_j = iub_j^*c_j + iu^*b_jc_j^* + i\frac{\epsilon_j}{2}(|c_j|^2 - |b_j|^2)$$
$$-i\frac{1}{2}\sum_{k \neq j} \frac{((|b_j|^2 - |c_j|^2)(|b_k|^2 - |c_k|^2) + 2b_j^*c_jb_kc_k^* + 2b_jc_j^*b_k^*c_k)}{\epsilon_j - \epsilon_k}, \tag{18}$$

$$J_0 = i|u|^2 + \frac{1}{2}i\sum_{j}^{n}(|b_j|^2 - |c_j|^2), \quad J_j = i(|b_j|^2 + |c_j|^2).$$

Next we develop a method which allows to construct quasi-periodic and periodic solutions of system (4-6). The method is based on the application of spectral theory for self-adjoint one dimensional Dirac equation with quasi-(periodic) finite gap potential $\mathcal{U} = -u$ cf. Eqs. (4,5)

$$\imath\frac{d\Psi_{1j}}{d\xi} - \mathcal{U}\Psi_{2j} - i\lambda_j\Psi_{1j} = 0, \tag{19}$$

$$\imath\frac{d\Psi_{2j}}{d\xi} - \mathcal{U}^*\Psi_{1j} + i\lambda_j\Psi_{1j} = 0, \tag{20}$$

with spectral parameter λ and eigenvalues $\lambda_j = i\epsilon_j/2$. The equation (7) is equivalently written as

$$\frac{dA}{d\xi} = iuC - iu^*B, \quad A(\xi, \lambda) = \sum_{j=0}^{n+1} A_{n+1-j}(\xi)\lambda^j, \tag{21}$$

$$\frac{dB}{d\xi} = -i\lambda B - 2iuA, \quad B(\xi, \lambda) = \sum_{j=0}^{n} B_{n-j}(\xi)\lambda^j, \tag{22}$$

$$\frac{dC}{d\xi} = i\lambda C + 2iu^*A, \quad C(\xi, \lambda) = \sum_{j=0}^{n} C_{n-j}(\xi)\lambda^j, \tag{23}$$

or in different form we have

$$A_{j+1,\xi} = iuC_j - iu^*B_j, \ A_0 = 1, A_1 = c_1, \tag{24}$$
$$iB_{j+1} = -B_{j,\xi} - 2iuA_{j+1}, \quad B_0 = -2u, \tag{25}$$
$$iC_{j+1} = C_{j,\xi} - 2iu^*A_{j+1} \quad C_0 = -2u^*, \tag{26}$$

where c_1 is the constant of integration. Differenciating Eq. (21) and using (15) we can obtain

$$BB_{\xi\xi} - \frac{u_\xi}{u}BB_\xi - \frac{1}{2}B_\xi^2 + \left(\frac{\lambda^2}{2} - i\lambda\frac{u_\xi}{u} + |u|^2\right)B^2 = 2u^2\nu. \tag{27}$$

Using (**??**) the eigenfunction Ψ_1 for finite-gap potential $\mathcal{U}$ have the form

$$\Psi_1(\xi, \lambda) = \left[\frac{\mathcal{U}(\xi)}{\mathcal{U}(0)}\prod_{j=1}^{n}\frac{\lambda - \mu_j(\xi)}{\lambda - \mu_j(0)}\right]^{1/2}\exp\left\{-i\int_0^\xi\frac{\sqrt{R(\lambda)}}{\prod_{j=1}^{n}(\lambda - \mu(\xi'))}d\xi'\right\}.$$

Special case of system (4-6) is the three wave system

$$i\frac{dA_1}{d\xi} = \epsilon A_3 A_2^*, \quad i\frac{dA_2}{d\xi} = \epsilon A_1^* A_3, \quad i\frac{dA_3}{d\xi} = \epsilon A_1 A_2. \tag{28}$$

The corresponding elements of Lax matrices are

$$L(\xi, \lambda) = \begin{pmatrix} -i\frac{\lambda}{2} + \frac{i}{2\lambda}\mathcal{A} & -i\epsilon A_1 - i\frac{1}{\lambda}A_3 A_2^* \\ -i\epsilon A_1^* - i\frac{1}{\lambda}A_2 A_3^{*2} & i\frac{\lambda}{2} - \frac{i}{2\lambda}\mathcal{A} \end{pmatrix}, \tag{29}$$

$$M(\xi, \lambda) = \begin{pmatrix} -i\frac{\lambda}{2} & -i\epsilon A_1 \\ -i\epsilon A_1^* & -i\lambda 2 \end{pmatrix}, \quad \mathcal{A} = |A_2|^2 - |A_3|^2. \tag{30}$$

To integrate the system (28) we introduce new variable

$$\mu = i\frac{1}{A_1}\frac{dA_1}{d\xi} = \frac{A_3 A_2^*}{A_1}, \tag{31}$$

in terms of which our equations can be written as

$$\frac{d\mu}{d\xi} = 2i\sqrt{R(\mu)}, \tag{32}$$

where

$$R(\lambda) = (\frac{1}{4}\lambda^2 - \frac{1}{2}\mathcal{A})^2 - |A_1|^2(\lambda - \mu)(\lambda - \mu^*) = \tag{33}$$

$$\frac{1}{4}\lambda^4 - \alpha_1\lambda^3 + \alpha_2\lambda^2 - \alpha_3\lambda + \alpha_4. \tag{34}$$

The μ variable and $A_j, j = 1, \ldots 3$ obey the equations

$$\alpha_1 = 0, \quad |A_1|^2 - \frac{1}{2}\mathcal{A} = \alpha_2, \tag{35}$$

$$-|A_1|^2(\mu + \mu^*) = \alpha_3, \quad \frac{1}{4}\mathcal{A} - \mu\mu^*|A_1|^2 = \alpha_4. \tag{36}$$

which are related to the integrals of motion of the monomer system with $\alpha_4 = 0$. The equation of motion is then

$$\left(\frac{d\mu}{d\xi}\right)^2 = -4\left(\mu^4 + N\mu^2 - H\mu\right) \tag{37}$$

where the system (28) conserves the dimensionless variable N and the Hamiltonian H

$$N = |A_1|^2 - \frac{1}{2}\mathcal{A}, \quad H = A_1 A_2 A_3^* + A_3 A_1^* A_2, \tag{38}$$

Solving Eqs. (36) for μ variable we obtain

$$\mu = \frac{1}{4\nu}\left(H + \imath\sqrt{P(\nu)}\right), \tag{39}$$

where

$$P(\nu) = 4\nu^3 - 4N\nu^2 + N^2\nu - H^2. \tag{40}$$

We seek the solution A_1 in the following form

$$A_1 = \sqrt{\nu(\xi)}\exp\left(iC\int_0^\xi \frac{d\xi'}{\nu(\xi')}\right) = \sqrt{\nu(\xi)}\exp(i\psi(\xi)), \tag{41}$$

where $\nu = |A_1|^2 = \wp(\xi + \omega') + C_1$, $\wp$ is the Weierstrass function, and ω' is half period. Using Eq. (39) and the following equation

$$\frac{d\nu}{d\xi} = -2\imath\nu(\mu - \mu^*), \tag{42}$$

derived from (31) and three wave equations we obtain

$$\left(\frac{d\nu}{d\xi}\right)^2 = 4\nu^3 - 4N\nu^2 + N^2\nu - H^2, \tag{43}$$

whose solution can be expressed in terms of the Weierstrass elliptic functions as

$$\nu = \wp(\xi + \omega') + \frac{N}{3}. \tag{44}$$

Substituting this expression in Eq. (41) we obtain

$$A_1 = \sqrt{\wp(\xi + \omega') + \frac{N}{3}}\exp(i\psi(\xi)), \tag{45}$$

where the phase $\psi(\xi)$ is given by

$$\psi(\xi) = \frac{H}{2\wp'(\kappa)}\left(\ln\frac{\sigma(\xi + \omega' - \kappa)}{\sigma(\xi + \omega' + \kappa)} + 2\zeta(\kappa)\xi\right) + \psi_0, \tag{46}$$

and ψ_0 is initial constant phase.

III.   BI-HAMILTONIAN STRUCTURE

In this paragraph we will compute r-matrix algebra of restricted multiple three wave interaction system. We note that in Lax representation (9) we remove the function $a(\lambda)$, which is essential for studying Hamiltonian dynamics of restricted multiple three wave interaction system.

Let as consider Lax matrix

$$\mathcal{L}(\lambda) = a^{-1}(\lambda)L(\xi, \lambda) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & -\mathcal{A} \end{pmatrix}. \tag{47}$$

and introduce standard Poisson bracket, $\{\cdot, \cdot\}_0$

$$\{f, g\}_0 = -\imath \left(\frac{\partial f}{\partial u} \frac{\partial g}{\partial u^*} - \frac{\partial f}{\partial u^*} \frac{\partial g}{\partial u} \right) - \imath \sum_{j=1}^n \left(\frac{\partial f}{\partial b_j} \frac{\partial g}{\partial b_j^*} - \frac{\partial f}{\partial b_j^*} \frac{\partial g}{\partial b_j} \right)$$

$$-\imath \left(\frac{\partial f}{\partial c_j} \frac{\partial g}{\partial c_j^*} - \frac{\partial f}{\partial c_j^*} \frac{\partial g}{\partial c_j} \right)$$

The entries of $\mathscr{L}$ satisfy to the following well known equations

$$\{\mathscr{A}(\lambda), \mathscr{A}(\mu)\}_0 = \{\mathscr{B}(\lambda), \mathscr{B}(\mu)\}_0 = \{\mathscr{C}(\lambda), \mathscr{C}(\mu)\}_0 = 0,$$

$$\{\mathscr{A}(\lambda), \mathscr{B}(\mu)\}_0 = \frac{1}{\lambda - \mu} \Big(\mathscr{B}(\mu) - \mathscr{B}(\lambda) \Big),$$

$$\{\mathscr{A}(\lambda), \mathscr{C}(\mu)\}_0 = \frac{-1}{\lambda - \mu} \Big(\mathscr{C}(\mu) - \mathscr{C}(\lambda) \Big),$$

$$\{\mathscr{B}(\lambda), \mathscr{C}(\mu)\}_0 = \frac{2}{\lambda - \mu} \Big(\mathscr{A}(\mu) - \mathscr{A}(\lambda) \Big),$$

which may be rewritten as linear r-matrix algebra

$$\{\overset{1}{\mathscr{L}}(\lambda), \overset{2}{\mathscr{L}}(\mu)\}_0 = [r(\lambda - \mu), \overset{1}{\mathscr{L}}(\lambda) + \overset{2}{\mathscr{L}}(\mu)],\tag{48}$$

here $\overset{1}{\mathscr{L}}(\lambda) = \mathscr{L}(\lambda) \otimes \mathrm{I}$, $\overset{2}{\mathscr{L}}(\mu) = \mathrm{I} \otimes \mathscr{L}(\mu)$ and $r(\lambda - \mu)$ is a classical rational r-matrix:

$$r(\lambda - \mu) = \frac{\Pi}{\lambda - \mu}, \qquad \Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remind, that two Poisson brackets $\{., .\}_0$ and $\{., .\}_1$ are compatible if every linear combination of them is still a Poisson bracket. The corresponding compatible Poisson tensors P_0 and P_1 satisfy to the following equations

$$[\![P_0, P_0]\!] = [\![P_0, P_1]\!] = [\![P_1, P_1]\!] = 0,\tag{49}$$

where $[\![., .]\!]$ is the Schouten bracket [6]. Remind that on a smooth finite-dimensional manifold $\mathscr{M}$ the Schouten bracket of two bivectors X and Y is an antisymmetric contravariant tensor of rank three and its components in local coordinates z_m read

$$[\![X, Y]\!]^{ijk} = -\sum_{m=1}^{\dim \mathscr{M}} \left(X^{mk} \frac{\partial Y^{ij}}{\partial z_m} + Y^{mk} \frac{\partial X^{ij}}{\partial z_m} + \mathrm{cycle}(i, j, k) \right).$$

The Poisson bracket associated with the Poisson bivector P is equal to

$$\{f(z), g(z)\} = \langle df, P\, dg \rangle = \sum_{i,k} P^{ik}(z) \frac{\partial f(z)}{\partial z_i} \frac{\partial g(z)}{\partial z_k}.\tag{50}$$

Here df is covector with entries $\partial f / \partial z_i$ and $\langle ., . \rangle$ is a standard vector product.

There are a lot of the Poisson brackets $\{., .\}_1$ compatible with the linear r-matrix bracket (48) similar to the quadratic Sklyanin algebra [7]. Here we consider two examples only.

Proposition 1 *If*

$$\mathscr{A} = \sum_{i=1}^n \frac{h_i}{\lambda - \epsilon_i}, \quad \mathscr{B} = 1 + \sum_{i=1}^n \frac{e_i}{\lambda - \epsilon_i}, \quad \mathscr{C} = \sum_{i=1}^n \frac{f_i}{\lambda - \epsilon_i},\tag{51}$$

where h_i, e_i, f_i are dynamical variables and ϵ_i are numerical parameters, then the following brackets are compatible with linear r-matrix bracket (48)

$$\{\mathscr{B}(\lambda), \mathscr{B}(\mu)\}_1 = \{\mathscr{A}(\lambda), \mathscr{A}(\mu)\}_1 = 0,$$

$$\{\mathscr{A}(\lambda), \mathscr{B}(\mu)\}_1 = \frac{1}{\lambda - \mu}\Big(\lambda\mathscr{B}(\mu) - \mu\mathscr{B}(\lambda)\Big) - \mathscr{B}(\lambda)\mathscr{B}(\mu),$$

$$\{\mathscr{A}(\lambda), \mathscr{C}(\mu)\}_1 = \frac{-\lambda}{\lambda - \mu}\Big(\mathscr{C}(\mu) - \mathscr{C}(\lambda)\Big) + \mathscr{B}(\lambda)\mathscr{C}(\mu), \tag{52}$$

$$\{\mathscr{B}(\lambda), \mathscr{C}(\mu)\}_1 = \frac{2}{\lambda - \mu}\Big(\mu\mathscr{A}(\mu) - \lambda\mathscr{A}(\lambda)\Big) + 2\Big(1 - \mathscr{B}(\lambda)\Big)\mathscr{A}(\mu),$$

$$\{\mathscr{C}(\lambda), \mathscr{C}(\mu)\}_1 = 2\Big(\mathscr{A}(\mu)\mathscr{C}(\lambda) - \mathscr{A}(\lambda)\mathscr{C}(\mu)\Big).$$

Proof: It is sufficient to check the statement on an open dense subset of the linear r-matrix algebra (48) defined by the assumption that all the h_i, e_i, f_i and ϵ_i are different. Namely, substituting rational functions (51) into (48) one gets canonical brackets on the direct sum of n copies of $sl(2)$

$$\{h_j, e_j\}_0 = e_j, \qquad \{h_j, f_j\}_0 = -f_j, \qquad \{e_j, f_j\}_0 = 2h_j, \qquad j = 1, \ldots, n. \tag{53}$$

Substituting these rational functions (51) into the second brackets (52) one gets the following non-local brackets between generators h_i, e_i, f_i

$$\{h_j, e_j\}_1 = (\epsilon_j - e_j)e_j, \quad \{h_j, f_j\}_1 = -(\epsilon_j - e_j)f_j, \quad \{e_j, f_j\}_1 = 2(\epsilon_j - e_j)h_j,$$

$$\{h_i, e_j\}_1 = -e_i b_j, \quad \{h_i, f_j\}_1 = e_i c_j, \quad \{e_i, f_j\}_1 = -2e_i a_j, \quad \{f_i, f_j\}_1 = -2h_i c_j + 2f_i a_j. \tag{54}$$

Now it is easy to prove that Poisson bracket (53) is compatible with the Poisson bracket (54).

Proposition 2 *If*

$$\mathscr{A} = h_n \lambda + \sum_{i=1}^{n-1} \frac{h_i}{\lambda - \epsilon_i}, \quad \mathscr{B} = e_n + \sum_{i=1}^{n-1} \frac{e_i}{\lambda - \epsilon_i}, \quad \mathscr{C} = f_n + \sum_{i=1}^{n-1} \frac{f_i}{\lambda - \epsilon_i} \tag{55}$$

where h_i, e_i, f_i are dynamical variables and ϵ_i are numerical parameters, then the following brackets are compatible with linear r-matrix bracket (48)

$$\{\mathscr{B}(\lambda), \mathscr{B}(\mu)\}_1 = \{\mathscr{A}(\lambda), \mathscr{A}(\mu)\}_1 = 0,$$

$$\{\mathscr{A}(\lambda), \mathscr{B}(\mu)\}_1 = \frac{1}{\lambda - \mu}\Big(\lambda\mathscr{B}(\mu) - \mu\mathscr{B}(\lambda)\Big) - \rho_1\mathscr{B}(\lambda)\mathscr{B}(\mu),$$

$$\{\mathscr{A}(\lambda), \mathscr{C}(\mu)\}_1 = \frac{-\lambda}{\lambda - \mu}\Big(\mathscr{C}(\mu) - \mathscr{C}(\lambda)\Big) + \rho_1\mathscr{B}(\lambda)\mathscr{C}(\mu) - \rho_2\mathscr{B}(\lambda), \tag{56}$$

$$\{\mathscr{B}(\lambda), \mathscr{C}(\mu)\} = \frac{2}{\lambda - \mu}\Big(\mu\mathscr{A}(\mu) - \lambda\mathscr{A}(\lambda)\Big) + 2\Big(1 - \rho_1\mathscr{B}(\lambda)\Big)\mathscr{A}(\mu) - \rho_3\,\mathscr{B}(\lambda)$$

$$\{\mathscr{C}(\lambda), \mathscr{C}(\mu)\} = -2\rho_1\Big(\mathscr{A}(\lambda)C(\mu) - \mathscr{A}(\mu)\mathscr{C}(\lambda)\Big) + 2\rho_2\Big(\mathscr{A}(\lambda) - \mathscr{A}(\mu)\Big) + \rho_3\Big(\mathscr{C}(\lambda) - \mathscr{C}(\mu)\Big)$$

Here

$$\rho_1 = \frac{1}{e_n} = \left[\frac{1}{\mathscr{B}(\lambda)}\right], \qquad \rho_2 = \frac{f_n}{e_n} = \left[\frac{C(\lambda)}{\mathscr{B}(\lambda)}\right], \tag{57}$$

and

$$\rho_3 = 1 - \frac{2h_n(\lambda + \mu)}{e_n} + \frac{2h_n \sum_{k=1}^{n-1} e_k}{e_n} = 1 - \left[\frac{(\lambda + \mu)\mathscr{A}(\lambda)}{\lambda\mathscr{B}(\lambda)}\right] - \left[\frac{(\lambda + \mu)\mathscr{A}(\mu)}{\mu\mathscr{B}(\mu)}\right], \tag{58}$$

where $[\mathscr{X}/\mathscr{Y}]$ is a quotient of polynomials X and Y in variables λ and μ over a field as in [7].

Proof: As above it is sufficient to check the statement on an open dense subset of the linear r-matrix algebra (48) defined by the assumption that all the h_i, e_i, f_i and ϵ_i are different.

Namely, substituting rational functions (51) into (48) one gets local brackets $n-1$ copies of $sl(2)$

$$\{h_j, e_j\}_0 = e_j, \qquad \{h_j, f_j\}_0 = -f_j, \qquad \{e_j, f_j\}_0 = 2h_j, \qquad j = 1, \ldots, n-1, \tag{59}$$

and degenerate brackets

$$\{e_n, f_n\} = -2h_n, \qquad \{h_n, h_i\} = \{h_n, e_i\} = \{h_n, f_i\} = 0. \tag{60}$$

The leading coefficients h_n is the Casimir element for these brackets.

Substituting rational functions (51) into the second brackets (56) one gets second non-local brackets between generators h_i, e_i, f_i. At $i, j = 1, \ldots, n-1$ these brackets looks like

$$\{h_j, e_j\}_1 = \left(\epsilon_j - \frac{e_j}{e_n}\right)e_j, \quad \{h_j, f_j\}_1 = -\left(\epsilon_j - \frac{e_j}{e_n}\right)f_j, \quad \{e_j, f_j\}_1 = 2\left(\epsilon_j - \frac{e_j}{e_n}\right)h_j,$$

$$\{h_i, e_j\}_1 = -\frac{e_i b_j}{e_n}, \quad \{h_i, f_j\}_1 = \frac{e_i c_j}{e_n}, \quad \{e_i, f_j\}_1 = -\frac{2e_i a_j}{e_n}, \quad \{f_i, f_j\}_1 = \frac{-2h_i c_j + 2f_i a_j}{e_n}. \tag{61}$$

At $e_n = 1$ these brackets coincide with the previous brackets (54).

The remaining non-zero brackets have the following form

$$\{f_i, f_n\}_1 = \rho_3(\lambda + \mu = \epsilon_j)f_i, \qquad \{e_i, f_n\}_1 = -\rho_3(\lambda + \mu = \epsilon_j)e_i, \qquad \{e_n, f_n\} = -e_n. \tag{62}$$

Now it is easy to prove that Poisson bracket (59)-(60) is compatible with the Poisson bracket (61)-(62). This completes the proof.

In the both cases we can rewrite second Poisson brackets in the following r-matrix form

$$\{\overset{1}{\mathscr{L}}(\lambda), \overset{2}{\mathscr{L}}(\mu)\}_1 = [r_{12}(\lambda, \mu), \overset{1}{T}(\lambda)] - [r_{21}(\lambda, \mu), \overset{2}{T}(\mu)], \tag{63}$$

where

$$r_{12}(\lambda, \mu) = \begin{pmatrix} \frac{\mu}{\lambda-\mu} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\lambda-\mu} & 0 \\ 0 & \frac{\lambda}{\lambda-\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\lambda-\mu} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_1 B(\mu) & 0 \\ -\rho_3 & 0 & 0 & 0 \\ \rho_2 - \rho_1 C(\mu) & 0 & 0 & 0 \end{pmatrix} \tag{64}$$

and

$$r_{21}(\lambda, \mu) = \Pi r_{12}(\mu, \lambda)\Pi.$$

For the Lax matrix $\mathscr{L}$ with entries (51) we have

$$\rho_1 = 1, \qquad \rho_2 = 0, \qquad \rho_3 = 0.$$

For the Lax matrix $\mathscr{L}$ with entries (55) functions ρ_k are given by (57)- (58).

It is easy to see that entries of the Lax matrix (47) have the form (55). So, we can use quadratic-linear algebra (63) in order to get bi-hamiltonian description of the restricted multiple three wave interaction system. The first part of the brackets between variables u, u^*, b_i, b_i^* and c_i, c_i^* may be directly restored from the brackets (61)-(62). The remaining part has to be obtained from the compatibility conditions (49).

As a result the non-zero brackets with variables u and u^* look like

$$\{u, u^*\}_1 = iu, \qquad \{u^*, b_j\}_1 = -\tfrac{i}{2}b_j\rho_3(\lambda + \mu = \epsilon_j), \qquad \{u^*, b_j^*\}_1 = \frac{i}{2}b_j^*\rho_3(\lambda + \mu = \epsilon_j),$$

$$\{u^*, c_j\}_1 = \tfrac{i}{2}c_j\rho_3(\lambda + \mu = \epsilon_j), \qquad \{u^*, c_j^*\}_1 = -\frac{i}{2}c_j^*\rho_3(\lambda + \mu = \epsilon_j).$$

The local brackets at $j = 1, \ldots, n$ are equal to

$$\{b_j, c_j\}_1 = \frac{ib_j^2}{2u}, \qquad , \{b_j, c_j^*\}_1 = \frac{i\epsilon_j b_j}{c_j}, \qquad \{b_j, b_j^*\}_1 = \frac{i(b_j c_j^* - 2u\epsilon_j)}{2u},$$

$$\{b_j^*, c_j\}_1 = \frac{-i(b_j b_j^* + c_j c_j^*)}{2u}, \qquad \{b_j^*, c_j^*\}_1 = \frac{-i(2b_j^* \epsilon_j u - c_j c^{*\,2}_{\ j})}{2c_j u}, \qquad \{c_j, c_j^*\}_1 = \frac{-ib_j c_j^*}{2u}.$$

The non-local brackets at $i \neq j$ read as

$$\{b_i, b_j^*\}_1 = -\frac{ib_i c_j^*}{2u}, \qquad \{b_i, c_j\}_1 = \frac{ib_i b_j}{2u}, \qquad \{c_i, c_j^*\}_1 = -\frac{ib_i c_j^*}{2u}, \qquad \{c_i^*, b_j^*\}_1 = -\frac{ic_i^* c_j^*}{2u},$$

$$\{c_i, c_j\}_1 = \frac{i(b_i c_j - b_j c_i)}{2u}, \qquad \{c_i, b_j^*\}_1 = \frac{i(c_i c_j^* + b_i b_j^*)}{2u}, \qquad \{b_i^*, b_j^*\}_1 = \frac{-(b_i^* c_j^* - c_i^* b_j^*)}{2u}.$$

Other brackets are equal to zero, for instance

$$\{b_i, b_j\}_1 = \{b_i, c_j^*\} = \{c_i^*, c_j^*\}_1 = 0.$$

Using r-matrix algebras (48) and (63) it is easy to prove that integrals of motion $I_j \in \{J_0, J_i, \ldots, J_n, K_1, \ldots, K_n\}$ (18) are in the bi-involution with respect to the brackets $\{.,.\}_0$ and $\{.,.\}_1$:

$$\{I_j, I_k\}_0 = \{I_j, I_k\}_1 = 0.$$

The one of the main problems is that the main characteristic of the model, such as equation of motion, the Lax matrices and integrals of motion are invariant with respect to conjugation

$$u \leftrightarrow u^*, \qquad b_j \leftrightarrow b_j^*, \qquad c_j \leftrightarrow c_j^*, \qquad (65)$$

but the second brackets $\{.,.\}_1$ do not invariant with respect to this transformation.

The second problem is that the second brackets $\{.,.\}_1$ are rational brackets and we have an obvious problem with quantization of these brackets.

According to [8] there are different bi-hamiltonian structures for a given integrable system. So, we could try to find another bi-hamiltonian description of the restricted multiple three wave interaction system, which allows as to avoid these problems.

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